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# Similarity solution of a semi-infinite fluid-driven fracture in a linear elastic solid

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**Abstract.** A similarity solution for a steadily moving semi-infinite fluid-driven fracture in an impermeable linear elastic solid is described in this note. The existence of a lag, of a priori unknown length, between the crack tip and the fluid front is explicitly taken into account. The asymptotic behavior of the solution at the tip is consistent with linear elastic fracture mechanics, whereas its asymptotic behaviour at infinity corresponds to a singular solution constructed under the assumption of zero fluid lag. A universal relation between fluid lag and fracture toughness is obtained. © Académie des Sciences/Elsevier, Paris

**hydraulic fracture / fracture mechanics**

***Solution autosemblable d'une fracture hydraulique semi-infinie se propageant dans un milieu élastique***

**Résumé.** *Cette note décrit la solution autosemblable du problème stationnaire d'une fracture semi-infinie se propageant à vitesse constante dans un milieu élastique imperméable, par injection d'un fluide sous pression. Cette étude considère l'existence d'une cavité entre le bout de la fissure et le front du fluide injecté, dont la longueur est a priori inconnue. La résolution de ce problème conduit à l'obtention d'une relation universelle entre la longueur de la cavité et la tenacité du milieu de propagation. © Académie des Sciences/Elsevier, Paris*

***fracture hydraulique / mécanique de la fracture***

## *Version française abrégée*

La solution en bout d'une fracture créée par injection d'un fluide sous pression (fracturation hydraulique) a déjà fait l'objet de nombreuses études [1–8]. La construction d'une telle solution est difficile, même si l'on se limite au cadre de la mécanique de la rupture fragile et aux équations de la lubrification pour l'écoulement du fluide dans la fracture. La difficulté majeure réside dans la prise en compte d'une cavité en tête de fissure (front fluide distinct de l'extrémité de la fissure, *figure 1*). Un examen des équations montre en effet que cette cavité doit exister pour prévenir l'existence d'une singularité de la pression du fluide, ainsi qu'une singularité des contraintes en tête de fissure, incompatible avec la singularité en racine carrée de la mécanique de la rupture fragile. La taille de cette cavité fait partie de la solution du problème.

**Note présentée par Huy Duong BUI.**

Dans le cadre de cette note, nous étudions le problème stationnaire d'une fissure semi-infinie se propageant à vitesse constante $V$, perpendiculairement à la contrainte de compression $\sigma_o$ à l'infini (*figure 1*). Le milieu de propagation est linéaire élastique et imperméable ; il est caractérisé par un module élastique de déformation plane $E'$ et par une tenacité $K_{Ic}$. Le fluide est incompressible et newtonien (viscosité $\mu$). L'analyse a pour but de déterminer la longueur $\lambda$ de la cavité en tête de fracture, ainsi que l'ouverture de la fissure $w$ et la pression $p$ du fluide injecté en fonction de la position $x$. On admet que la pression $p_\lambda$ dans la cavité est négligeable ($p_\lambda = 0$).

Ces hypothèses conduisent à la formulation d'un système d'équations non linéaires, comprenant les équations différentielles (1), l'équation intégrale de Cauchy (2), et le critère de propagation (3). Un groupe de variables adimensionnelles est introduit après avoir défini (4), deux échelles de longueur $L_h$ et $L_k$ (respectivement associées à la dissipation dans le fluide et dans le solide) et un petit paramètre $\varepsilon$. Ces nouvelles variables sont l'ouverture $\Omega$, la pression nette du fluide $\Pi$ et la position $\xi$ dans un repère mobile dont l'origine coïncide avec la pointe de la fissure (*figure 1*). Après introduction de ces variables adimensionnelles, le système d'équations et de conditions aux limites (5)–(8), ne dépend plus que d'un nombre $\kappa$, pouvant être interprété comme une tenacité adimensionnelle (9). La longueur (adimensionnelle) $\Lambda$ de la cavité (9) ne dépend donc plus que de ce seul paramètre $\kappa$.

Par ailleurs, on peut montrer [9] que le comportement de la solution à l'infini correspond à la solution singulière (4), obtenue en supposant que le fluide pénètre jusqu'au bout d'une fracture se propageant dans un solide sans tenacité — solution dénotée ZTSS [7]. Les comportements asymptotiques à l'origine, (3) et (8), et à l'infini, (10) sont dès lors connus. La transition entre ces deux asymptotes ainsi que la longueur de la cavité sont calculées numériquement.

L'élément clé de la méthode de solution est l'utilisation de la formule d'inversion (13) de l'intégrale de Cauchy (7). Cette formule s'applique dans le cas où la fonction $\Pi$ se comporte comme $\xi^{-\alpha}$, où $\alpha > 0$ [11]. Elle est donc d'application plus générale que la formule d'inversion habituellement utilisée dans la résolution de problèmes impliquant une fracture semi-infinie dans un solide linéaire élastique ($\alpha \geq 1$). L'intégrale sur l'intervalle semi-infini $[0,\infty[$ apparaissant dans (13) est ensuite traitée comme une somme d'un terme connu correspondant au domaine $[0, \Lambda]$ et d'une intégrale sur $[\Lambda, \infty[$. L'intégration depuis l'origine de (13) sur $\xi$, en utilisant la condition d'ouverture nulle à l'origine, conduit à l'équation (14). La constante inconnue $C$ peut être exprimée en fonction de $\kappa$ et $\Lambda$, (17). Une méthode numérique est utilisée pour résoudre le système (5), (14)–(16), en tenant compte de la condition (17) ainsi que des conditions aux frontières (20). Pour le calcul, la longueur $\Lambda$ est imposée, et l'on en déduit le paramètre $\kappa(\Lambda)$.

La relation $\kappa(\Lambda)$ dérivée numériquement est illustrée par la *figure 2* (*tableau I*, également). La longueur $\Lambda$ de la cavité diminue quand la tenacité s'accroît ; la longueur maximale $\Lambda_o$ (atteinte pour une tenacité $\kappa$ nulle) est estimée à 0,357. Les variations de la pression du fluide $\Pi$ et de l'ouverture de la fissure $\Omega$ en fonction de la variable $\xi$ sont présentées sur les *figures 3* et *4*. La *figure 4* illustre bien les comportements asymptotiques de $\Omega$, au voisinage de l'origine ($\Omega = \kappa\xi^{1/2}$, sauf pour $\kappa = 0$ où $\Omega \sim \xi^{3/2}$) et à l'infini ($\Omega \sim \xi^{2/3}$).

La longueur $\varepsilon^{-3} L_h$ caractérise les phénomènes en bout de fracture dans la solution présentée. Habituellement, cette longueur caractéristique est de plusieurs ordres de grandeur plus petites que la longueur d'une fracture hydraulique. Ce rapport de longueur suggère que la solution décrite dans cette note pourrait en fait décrire le comportement en bout d'une fracture de longueur finie.

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## 1. Introduction

The development of a solution in the near tip region of fluid-driven fractures has been the subject of research effort for many years [1]–[8]. The difficulties that need to be resolved in constructing such a solution, are the presence of a lag of a priori unknown length between the fluid front and the crack tip and the existence of a very large fluid pressure gradient in the tip region; both affect the overall solution.

A solution for a semi-infinite fluid-driven fracture, propagating steadily in an impermeable linear elastic solid has recently been published [7]. This solution, built on the assumption that the fluid reaches the tip of the fracture, is characterized by a cubic root singularity in the stress field at the crack tip (for a Newtonian fluid). Such a result is in apparent contradiction with the stronger square root singularity predicted by linear elastic fracture mechanics (LEFM). Thus, this solution appears to be compatible only with a solid of zero toughness. (It will henceforth be referred to as the zero toughness singular solution, or ZTSS.) Moreover, the fluid pressure has a negative cubic root singularity (for a Newtonian fluid) at the tip. These paradoxical results (zero energy release rate and negative infinite fluid pressure at the tip) actually suggest the existence of a lag between the fluid front and the tip of the moving fracture.

The focus of the paper is on constructing the solution for a semi-infinite hydraulic crack, which accounts for the presence of a fluid lag of a priori unknown length. It is shown that this new solution is not only consistent with LEFM, but that its asymptotic behaviour at infinity is actually given by the ZTSS.

## 2. Governing equations

Consider a semi-infinite hydraulic crack, parallel to the fixed reference $x$-axis, propagating at constant velocity $V$ in an impermeable linear elastic medium (see *figure 1*). The crack is loaded by the internal fluid pressure $p(x,t)$ and by the far-field confining stress $\sigma_o$. A fluid lag (tip cavity) of length $\lambda$ is allowed adjacent to the crack tip. The length $\lambda$ of this cavity is unknown and is part of the solution. The tip cavity is assumed to be filled by fluid vapors under constant pressure $p_\lambda \approx 0$ ($p_\lambda$ is assumed to be negligible compared to the reference far-field stress $\sigma_o$). Since the crack propagation is stationary, the fracturing fluid front propagates with the same velocity as the crack tip $V$.

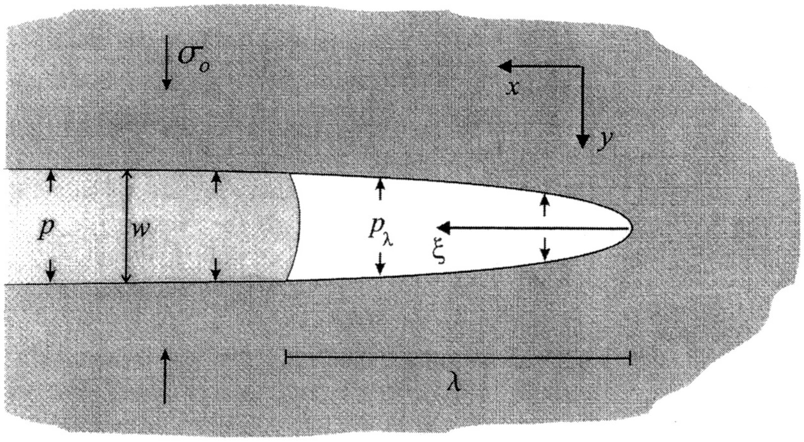


**Figure 1.** Semi-infinite fluid driven crack with the lag zone adjacent to the tip.

**Figure 1.** *Fracture hydraulique semi-infinie avec une cavité en bout de fracture.*

The set of governing equations is given by the equations of fluid flow from lubrication theory (continuity equation and Poiseuille law), the integral equation from the elasticity theory and the criterion for crack propagation

$$\frac{\partial w}{\partial t}+\frac{\partial q}{\partial x} = 0, \qquad q=-\frac{w^3}{12\,\mu}\frac{\partial p}{\partial x} \tag{1}$$

$$p(x,t)-\sigma_o = -\frac{E'}{4\pi}\int_{x_{\rm tip}}^{\infty}\frac{\partial w(s,t)}{\partial s}\frac{ds}{s-x} \tag{2}$$

$$K_I = K_{Ic} \tag{3}$$

where $w$ is the crack opening, $q$ the fluid flow rate, $E'$ the plane strain elastic modulus of the solid, $K_I$ and $K_{Ic}$ the stress intensity factor and the material toughness, respectively, and $\mu$ the dynamic viscosity of the fluid. Also, $x_{\rm tip}=-Vt$ denotes the tip position.

## 3. Scaling and dimensionless formulation

First, we introduce two lengthscales $L_h$ and $L_k$, and a small parameter $\varepsilon$ defined as follows

$$L_h=\frac{12\,\mu V}{E'},\qquad L_k=\frac{8}{\pi}\left(\frac{K_{Ic}}{E'}\right)^2,\qquad \varepsilon=\frac{\sigma_o}{E'} \tag{4}$$

The lengthscale $L_h$ is associated with viscous dissipation [7], while $L_k$ characterizes the dissipation due to fracturing of the solid. The following scaled quantities are then defined: the crack opening $\Omega=\varepsilon^2 w/L_h$, the moving coordinate $\xi=\varepsilon^3(x+Vt)/L_h$, and the net pressure $\Pi=(p-\sigma_o)/\sigma_o$. After performing the transformation to the moving coordinate $\xi$ and making use of the stationarity condition, the system (1)–(3) takes the dimensionless form

$$\Omega^2(\xi)\,\Pi'(\xi)=1 \qquad \text{for}\quad \xi\in(\Lambda,\infty) \tag{5}$$

$$\Pi=-1 \qquad \text{for}\quad \xi\in(0,\Lambda) \tag{6}$$

$$\Pi(\xi)=\frac{1}{4\pi}\int_0^{\infty}\Omega'(\eta)\frac{d\eta}{\xi-\eta} \tag{7}$$

$$\Omega=\kappa\sqrt{\xi}+O(\xi^{3/2}) \tag{8}$$

where equation (6) expresses the pressure condition in the lag region. In the above, $\Lambda$ denotes the dimensionless lag, and $\kappa$ the dimensionless toughness, respectively defined as

$$\Lambda=\frac{\varepsilon^3\lambda}{L_h},\qquad \kappa=2\sqrt{\frac{\varepsilon L_k}{L_h}} \tag{9}$$

Equation (8) is the LEFM asymptotic expression for the opening near the crack tip [10], with the propagation criterion (3) taken into account. Note that only one parameter $\kappa$ is present in the normalized system of equations and boundary conditions (5)–(8). This system completely defines the crack openin $\Omega(\xi;\kappa)$ and the net pressure $\Pi(\xi;\kappa)$ for the semi-infinite fracture $(0\le\xi<\infty)$, as well as the position of the fluid front $\Lambda(\kappa)$.

## 4. Asymptotic behavior at infinity

In the zero toughness singular solution [7], the fluid is assumed to flow to the crack tip. Hence, the lubrication equation (5) is valid along the whole crack length, $\xi\in(0,\infty)$, and no boundary condition is imposed on the net pressure at the tip, $\Pi(0)$. Enforcing equation (5) near the tip, together with the condition $\Omega(0) = 0$, necessarily yields a singularity in the fluid pressure at the tip. Actually, the requirement of a matching singularity between the lubrication (5) and elasticity equations (7) uniquely prescribes the form of this singularity as well as the whole self-similar solution, which is given by [7]

$$\Omega_\infty(\xi) = \frac{1}{2\sqrt{3}}(36\,\xi)^{2/3}, \qquad \Pi_\infty(\xi) = -\frac{1}{(36\,\xi)^{1/3}} \tag{10}$$

The ZTSS (10) has a weaker singularity than the one (8) predicted by LEFM; consistency of equation (10) requires, therefore, that the toughness $K_{Ic}$ be zero (or $\kappa = 0$). It is important to note that equation (10) cannot be the solution of the system (5)–(8) as the fluid lag goes to zero $\Lambda \to 0$. Indeed, the net pressure $\Pi$ is singular at the tip according to equation (10), whereas the solution $\Pi$ of the system (5)–(8) is finite at the tip in view of the boundary condition (6). It can be proven, however, that the ZTSS gives the asymptotic behavior of the solution of (5)–(8) at infinity [9]

$$\Pi(\xi) = \Pi_\infty(\xi) + O(\xi^{-1/3-\alpha}), \quad \alpha > 0, \quad \text{as} \quad \xi \to \infty \tag{11}$$

$$\Omega(\xi) = \Omega_\infty(\xi) + O(\xi^{1/2}), \quad \text{as} \quad \xi \to \infty \tag{12}$$

This asymptote does not depend on the toughness $\kappa$.

## 5. Solution

From the above considerations, the unknown solution behaves according to LEFM in the near tip region, see equations (6) and (8), and asymptotically as the ZTSS at large enough distance from the tip. Hence, there exists a transition zone in the solution between these two asymptotes. This intermediate solution has to be obtained numerically. The following considerations provide the basis of the algorithm used to compute this solution.

The Cauchy convolution integral (7) on the semi-infinite interval $\xi \in [0, \infty[$ has the inverse given by [11]

$$\Omega'(\xi) = \frac{C}{2\,\xi^{1/2}} - \frac{4}{\pi\xi^{1/2}} \int_0^\infty \frac{\xi + \frac{1}{2}}{\eta + \frac{1}{2}} \frac{\eta^{1/2}\,\Pi(\eta)}{\xi - \eta}\, d\eta \tag{13}$$

where $C$ is an arbitrary constant and the integral in equation (13) is taken in the sense of a Cauchy principal value. To ensure the existence of this integral, $\Pi(\xi)$ must behave at infinity as $\xi^{-\alpha}$ with $\alpha > 0$ (as is indeed the case in view of equation (11)). This prerequisite is weaker, however, than the one ($\alpha \geq 1$) for the classic inversion formula used in the solution of semi-infinite crack problems in LEFM [10].

Since the net loading is known in the lag zone, the integral in equation (13) can be split into a known contribution from the interval $[0, \Lambda]$ and an integral on $[\Lambda, \infty[$. Another integration over $\xi$ with the condition of zero opening at the tip yields

$$\Omega(\xi) = C\xi^{1/2} + \Omega_\Lambda(\xi) + \Omega_r(\xi) \tag{14}$$

where

$$\Omega_\Lambda(\xi) = \frac{4}{\pi}\left[(\xi - \Lambda)\ln\left|\frac{\sqrt{\xi} + \sqrt{\Lambda}}{\sqrt{\xi} - \sqrt{\Lambda}}\right| + 2\left[\sqrt{\Lambda} - \sqrt{2}\arctan\left(\sqrt{2\Lambda}\right)\right]\sqrt{\xi}\right] \tag{15}$$

$$\Omega_r(\xi) = \frac{4}{\pi}\int_\Lambda^\infty \left(\ln\left|\frac{\sqrt{\xi} + \sqrt{\eta}}{\sqrt{\xi} - \sqrt{\eta}}\right| - \frac{2\sqrt{\xi\eta}}{\eta + \frac{1}{2}}\right)\Pi(\eta)\, d\eta \tag{16}$$

The constant $C$ is part of the solution of the problem; it can be expressed in terms of the fluid lag $\Lambda$ and the toughness $\kappa$ using (8) and (14)–(16)

$$C = \kappa - \kappa_\Lambda - \kappa_r \qquad (17)$$

where

$$\kappa_\Lambda \equiv 2 \lim_{\xi \to 0} \xi^{1/2} \Omega'_\Lambda(\xi) = -\frac{8\sqrt{2}}{\pi} \arctan(\sqrt{2\Lambda}) \qquad (18)$$

$$\kappa_r \equiv 2 \lim_{\xi \to 0} \xi^{1/2} \Omega'_r(\xi) = \frac{4}{\pi} \int_\Lambda^\infty \frac{\Pi(\eta)\,\mathrm{d}\eta}{\eta^{1/2}\left(\eta + \frac{1}{2}\right)} \qquad (19)$$

It follows from the above considerations that the solution is now reduced to finding the lag length $\Lambda(\kappa)$, the net pressure $\Pi(\xi;\kappa)$ and the crack opening $\Omega(\xi;\kappa)$ along the semi-infinite interval $\xi \in (\Lambda, \infty)$. The solution must satisfy (5), (14)–(16) with (17) and the boundary conditions

$$\Pi(\Lambda) = -1, \quad \Pi(\infty) = 0 \qquad (20)$$

The solution of this boundary value problem is obtained numerically, using a boundary element approach which takes into account the asymptotic behavior (10) of the solution at infinity [9]. Note that the lag length $\Lambda$ is prescribed in the numerical solution, and the corresponding $\kappa = \kappa(\Lambda)$ computed from (17).

## 6. Numerical results

The computed variation of the stress intensity factor $\kappa$ with the fluid lag $\Lambda$ is plotted in *figure 2* in semi-log axes (see also *table I*). The fluid lag can be seen to increase with decreasing toughness, to reach a maximum value $\Lambda_o \simeq 0.3574$ for $\kappa = 0$. Note that this value of $\Lambda_o$ is very close to the value computed by Lister [5] for the problem of a buoyancy-driven hydraulic fracture, using a perturbation technique.

**Table I.** Table of corresponding values of the pair $(\kappa, \Lambda)$.

**Tableau I.** *Table des valeurs correspondantes de la paire* $(\kappa, \Lambda)$.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\kappa$ | 0 | 0.016 | 0.075 | 0.16 | 0.32 | 0.62 | 0.98 | 1.37 | 2.08 | 2.78 | 3.3 | 3.72 |
| $\Lambda$ | 0.3574 | 0.35 | 0.33 | 0.3 | 0.25 | 0.17 | 0.1 | 0.05 | 0.01 | 1E-3 | 1E-4 | 1E-5 |

It is of interest to compute the maximum dimension of the lag, $\lambda_o = \Lambda_o \varepsilon^{-3} L_h$, for some typical values of the physical parameters. Consider the following set: $E' = 3 \cdot 10^4$ MPa, $\mu = 10^{-7}$ MPa·s (100 cp), $\sigma_o = 10$ MPa, and $V = 1$ m·s$^{-1}$. Then, the characteristic length $\varepsilon^{-3} L_h = 1.08$ m and $\lambda_o \simeq 0.39$ m. The fluid lag reduces to $\lambda \simeq 0.27$ m for a toughness $K_{Ic} = 1$ MPa·m$^{1/2}$, according to *figure 2* ($\Lambda \simeq 0.25$ for $\kappa = 0.31$).

Plots of the net pressure $\Pi$ and crack opening $\Omega$ along the crack are shown in *figures 3* and *4*, for different values of $\kappa$ (and thus of $\Lambda$). It can be seen from the net pressure profiles (see *figure 3*), that the pressure increases rapidly from its value $\Pi = -1$ at the fluid front $\xi = \Lambda$, and that is approximately given by its asymptote at infinity (ZTSS) for $\xi > 100$. The log–log plot of the crack opening profile for various values of the toughness, shown in *figure 4*, provides transparent evidence that the solution behaves as $\xi^{1/2}$ (but as $\xi^{3/2}$ for $\kappa = 0$) in the region immediately adjacent to the tip

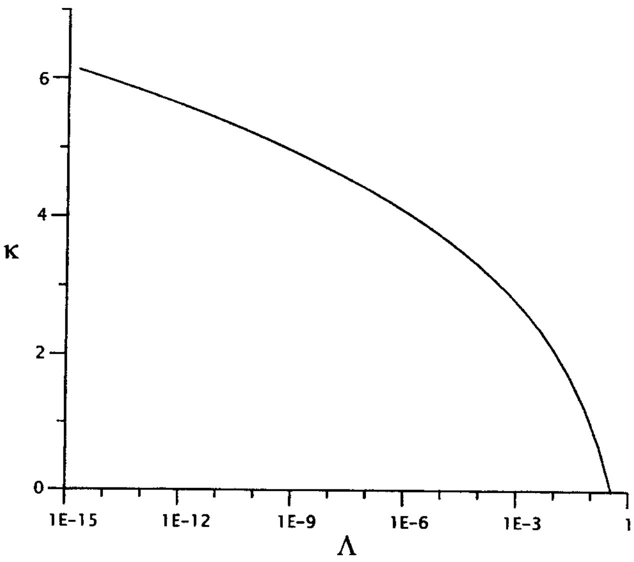


**Figure 2.** Dimensionless toughness $\kappa$ vs dimensionless lag length $\Lambda$ in semi-log scale.

**Figure 2.** *Variation de la tenacité adimensionnelle $\kappa$ avec la longueur adimensionnelle $\Lambda$ de la cavité, dans un système d'axes semi-logarihmiques.*

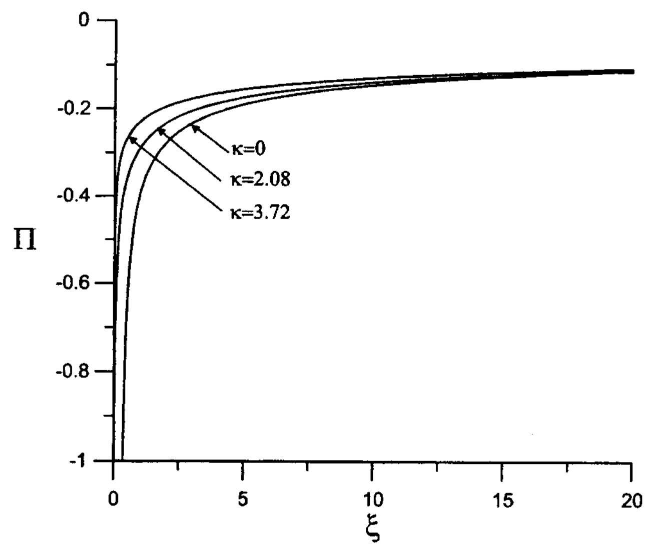


**Figure 3.** Dimensionless net pressure $\Pi$ along the crack for $\kappa = 0, 2.08, 3.72$.

**Figure 3.** *Variation de la pression adimensionnelle $\Pi$ le long de la fracture, pour* $\kappa = 0, 2{,}08, 3{,}72$.

(in accordance to LEFM), and that it converges towards the ZTSS $\Omega_\infty(\xi)$, shown in dashed line, further away from the tip. There is a transition zone between these two behaviours.

The numerical solution indicates that with increasing toughness $\kappa$, the fluid lag decreases while the region dominated by the LEFM $\xi^{1/2}$ behavior increases. Also, the transition to the asymptotic behavior at infinity occurs further away from the crack tip, as $\kappa$ increases. For small enough values of the toughness (large fluid lag), the LEFM region lies inside the lag entirely, whilst for large enough toughness (small lag) the LEFM region and the transition zone exceed the lag many times.

## 7. Concluding remarks

In this paper, we have outlined the solution of a semi-infinite fluid-driven fracture propagating steadily in an impermeable elastic solid of arbitrary toughness. The particularity of this solution is that

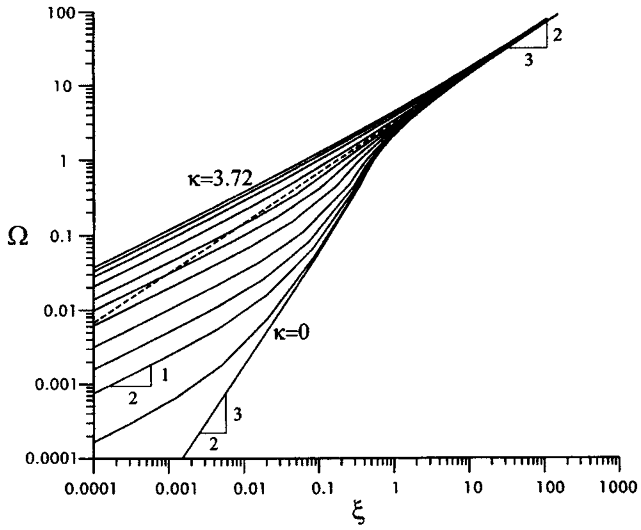


**Figure 4.** The dimensionless crack opening $\Omega$ along the crack in log–log scale for dimensionless toughness varying from $\kappa = 0$ ($\Lambda \simeq 0.357$) to $\kappa = 3.7$ ($\Lambda = 10^{-5}$). The dashed line corresponds to the asymptotic solution at infinity (ZTSS).

**Figure 4.** *Variation de l'ouverture (adimensionnelle) $\Omega$ le long de la fracture, pour une tenacité adimensionnelle variant entre $\kappa = 0$ ($\Lambda \simeq 0,357$) et $\kappa = 3,7$ ($\Lambda = 10^{-5}$). La ligne en tiret correspond à la solution asymptotique à l'infini (ZTSS).*

it accounts for the existence of a fluid lag, of a priori unknown length. The existence of this lag (where the pressure is essentially zero) allows the construction of a solution which has a near crack tip behavior consistent with LEFM. Indeed, the assumption that the fluid reaches the tip of the fracture, as in the ZTSS, implies a singularity in the fluid pressure and a crack tip behavior which is incompatible with LEFM. The ZTSS was shown, however, to correspond to the asymptotic behavior at infinity. Finally, it was demonstrated that the solution, and in particular the fluid lag length, depends only on the dimensionless toughness $\kappa$, which is an aggregate of all the parameters of the problem.

The characteristic length of the near tip processes, $\varepsilon^{-3} L_h$, is typically several orders of magnitude smaller than the length of hydraulic fractures (10–$10^3$ m). This difference in scale suggests that this solution (of a semi-infinite fracture) would actually describe the near tip asymptotic solution of a finite hydraulic fracture.

## References


[1] Kristianovic S.A., Zheltov, Formation of vertical fractures by means of highly viscous fluids, Proc. 4th World Petroleum Congress, Vol. II, Rome (1955) 579–586.

[2] Geertsma, Klerk F. De, A rapid method of predicting width and extent of hydraulic induced fractures, JPT 246 (1969) 1571–1581.

[3] Bui H.D., Parnes R., A reexamination of the pressure at the tip of a fluid-filled crack, Int. J. Engng. Sci. 20 (11) (1982) 1215–1220.

[4] Medlin W.L., Masse L., Laboratory experiments in fracture propagation, Soc. Pet. Eng. J. (1984) 256–268.

[5] Lister J.R., Bouyancy-driven fluid fracture: the effects of material toughness and of low-viscosity precursors, J. Fluid Mech. 210 (1990) 263–280.

[6] Rubin A.M., Tensile fracture of rock at high confining pressure: implications for dike propagation, J. Geophy. Res. 98 (1993) 15919–15935.

[7] Desroches J., Detournay E., Lenoach B., Papanastasiou P., Pearson J.R.A., Thiercelin M., Cheng A. H-D., The crack tip region in hydraulic fracturing, Phil. Trans. Roy. Soc. London, Ser. A, A447 (1994) 39–48.

[8] Advani S.H., Lee T.S., Dean R.H., Pak C.K., Avasthi J.M., Consequences of fluid lag in three-dimensional hydraulic fractures, Int. J. Numer. Anal. Methods Geomech. 21 (1997) 229–240.

[9] Garagash D., Detournay E., The crack tip region of a fluid-driven fracture propagating in an impermeable elastic medium, ASME J. Appl. Mech. (1998) (to be submitted).

[10] Rice J.R., Mathematical analysis in the mechanics of fracture, Fracture, an Advanced Treatise, vol. II, Academic Press, 1968, ch. 3, pp. 191–311.

[11] Srivastava H.M., Buschman R.G., Theory and applications of convolution integral equations, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1992, 240 pp.